

SHARP POWER MEANS BOUNDS FOR NEUMAN-SÁNDOR MEAN

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ABSTRACT. For $a, b > 0$ with $a \neq b$, let $N(a, b)$ denote the Neuman-Sándor mean defined by

$$N(a, b) = \frac{a - b}{2 \operatorname{arcsinh} \frac{a-b}{a+b}}$$

and $A_r(a, b)$ denote the r -order power mean. We present the sharp power means bounds for the Neuman-Sándor mean:

$$A_{p_1}(a, b) < N(a, b) \leq A_{p_2}(a, b),$$

where $p_1 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$ and $p_2 = 4/3$ are the best constants.

1. INTRODUCTION

Throughout the paper, we assume that $a, b > 0$ with $a \neq b$. The classical power mean of order r of the positive real numbers a and b is defined by

$$A_r = A_r(a, b) = \left(\frac{a^r + b^r}{2} \right)^{1/r} \quad \text{if } r \neq 0 \text{ and } A_0 = A_0(a, b) = \sqrt{ab}.$$

It is well-known that the function $r \mapsto A_r(a, b)$ is continuous and strictly increasing on \mathbb{R} (see [2]). As special cases, the arithmetic mean, geometric mean and quadratic mean are $A = A(a, b) = A_1(a, b)$, $G = G(a, b) = A_0(a, b)$ and $Q = Q(a, b) = A_2(a, b)$, respectively.

The logarithmic mean and identric (exponential) mean are defined as

$$\begin{aligned} L &= L(a, b) = \frac{a - b}{\ln a - \ln b}, \\ I &= I(a, b) = e^{-1} (a^a / b^b)^{1/(a-b)}, \end{aligned}$$

respectively. In 1993, Seiffert [19] introduced his first mean as

$$(1.1) \quad P = P(a, b) = \frac{a - b}{4 \arctan \sqrt{a/b} - \pi},$$

which can be written also in the equivalent form

$$(1.2) \quad P = P(a, b) = \frac{a - b}{2 \arcsin \frac{a-b}{a+b}},$$

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see e.g. [18]. In 1995, Seiffert [20] defined his second mean as

$$T = T(a, b) = \frac{a - b}{2 \arctan \frac{a-b}{a+b}}.$$

Recently, Neuman and Sándor have defined in [12] a new mean

$$(1.3) \quad N = N(a, b) = \frac{a - b}{2 \operatorname{arcsinh} \frac{a-b}{a+b}} = \frac{a - b}{2 \ln \frac{a-b + \sqrt{2(a^2+b^2)}}{a+b}}$$

All these means are symmetric and homogeneous, and the power mean is relatively simple. Hence ones are interested in evaluating these means by power means A_p .

Ostle and Terwilliger [13] and Karamata [7] first proved that

$$(1.4) \quad G < L < A.$$

This result, or a part of it, has been rediscovered and reproved many times (see e. g., [10], [23], [24], [15]). In 1974 Lin [9] obtained an important refinement of the above inequalities:

$$(1.5) \quad G < L < A_{1/3},$$

and proved that the number $1/3$ cannot be replaced by a smaller one.

For the identric mean I , Stolarsky [21] first proved that

$$(1.6) \quad G < I < A$$

(also see [23], [24]). In 1988, Alzer [1] showed that

$$(1.7) \quad 2e^{-1}A < I < A$$

(also see [16]). The following double inequality

$$(1.8) \quad A_{1/2} < I < 4e^{-1}A_{1/2}$$

is due to Neuman and Sándor [11]. Stolarsky [22] and Pittenger [14] established the sharp lower and upper bounds for I in terms of power means

$$(1.9) \quad A_{2/3} < I < A_{\ln 2},$$

respectively. By using the well properties of homogeneous functions, Yang also proved (1.7), (1.8) in [25] and

$$(1.10) \quad A_{2/3} < I < 2\sqrt{2}e^{-1}A_{2/3}$$

in [26].

For the first Seiffert mean P , the author [19] gave a estimate by A

$$(1.11) \quad \frac{2}{\pi}A < P < A.$$

Subsequently, Jagers [6] proved that

$$(1.12) \quad A_{1/2} < P < A_{2/3}.$$

By using Pfaff's algorithm Sádor in [17] reproved the first inequality in (1.12), while Hästö [4] gave a companion one of the second one in (1.12):

$$(1.13) \quad \frac{2\sqrt{2}}{\pi}A_{2/3} < P < A_{2/3},$$

Two year later, Hästö [5] obtained further a sharp lower bound for P :

$$(1.14) \quad P > A_{\ln_\pi 2}.$$

In 1995, Seiffert [20] showed that

$$(1.15) \quad A < T < A_2.$$

Very recently, Yang [27] present the sharp bounds for the second Seiffert mean in terms of power means:

$$(1.16) \quad A_{\log_{\pi/2} 2} < T \leq A_{5/3}.$$

Moreover, he obtained that

$$(1.17) \quad \alpha A_{5/3} < T < A_{5/3},$$

$$(1.18) \quad A_{\log_{\pi/2} 2} < T < \beta A_{\log_{\pi/2} 2},$$

where $\alpha = 2^{8/5} \pi^{-1} = 0.96494\dots$ and $\beta = 1.5349\dots$ are the best possible constants.

Concerning the Neuman-Sándor mean, the author [12] first established

$$(1.19) \quad G < L < P < A < N < T < A_2$$

and

$$(1.20) \quad \frac{\pi}{2} P > A > \operatorname{arcsinh}(1) N > \frac{\pi}{2} T.$$

Lately, Costin and Toader [3, Theorem 1] have shown that $A_{3/2}$ can be put between N and T , that is,

$$(1.21) \quad N < A_{3/2} < T,$$

and they obtained the following nice chain of inequalities for certain means:

$$(1.22) \quad G < L < A_{1/2} < P < A < N < A_{3/2} < T < A_2.$$

Our aim is to prove that

$$(1.23) \quad A_{\frac{\ln 2}{\ln \ln(3+2\sqrt{2})}} < N < A_{4/3},$$

where $\frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$ and $4/3$ are the best possible constants. Thus, we obtain a more nice chain of inequalities for bivariate means:

$$\begin{aligned} A_0 &< L < A_{1/3} < A_{\ln_\pi 2} < P < A_{2/3} < I < A_{\ln 2} \\ &< A_{\frac{\ln 2}{\ln \ln(3+2\sqrt{2})}} < N < A_{4/3} < A_{\log_{\pi/2} 2} < T < A_{5/3} \end{aligned}$$

Our main results are the following

Theorem 1. *For $a, b > 0$ with $a \neq b$, the inequality $N < A_p$ holds if and only if $p \geq 4/3$. Moreover, we have*

$$(1.24) \quad \alpha_1 A_{4/3} < N < \beta_1 A_{4/3},$$

where $\alpha_1 = \frac{1}{\sqrt[4]{2} \ln(\sqrt{2}+1)} = 0.95407\dots$ and $\beta_1 = 1$ are the best possible constants.

Theorem 2. *For $a, b > 0$ with $a \neq b$, the inequality $N > A_p$ holds if and only if $p \leq p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \approx 1.2228$. Moreover, we have*

$$(1.25) \quad \alpha_2 A_{p_0} < N < \beta_2 A_{p_0},$$

where $\alpha_2 = 1$ and $\beta_2 \approx 1.0138$ are the best possible constants.

2. LEMMAS

In order to prove our main results, we need the following lemmas.

Lemma 1. *Let F_p be the function defined on $(0, 1)$ by*

$$(2.1) \quad F_p(x) = \ln \frac{N(1, x)}{A_p(1, x)} = \ln \frac{x-1}{2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} - \frac{1}{p} \ln \left(\frac{x^p+1}{2} \right).$$

Then we have

$$(2.2) \quad \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-4),$$

$$(2.3) \quad F_p(0^+) = \lim_{x \rightarrow 0^+} F_p(x) = \begin{cases} \frac{1}{p} \ln 2 - \ln \ln(3+2\sqrt{2}) & \text{if } p > 0, \\ \infty & \text{if } p \leq 0, \end{cases}$$

where $F_0(x) := \lim_{p \rightarrow 0} F_p(x)$.

Proof. Using power series expansion we have

$$F_p(x) = -\frac{1}{24}(3p-4)(x-1)^2 + O((x-1)^3),$$

which yields (2.2).

Direct limit calculation leads to (2.3), which proves the lemma. \square

Lemma 2. *Let F_p be the function defined on $(0, 1)$ by (2.1). Then F_p is strictly increasing on $(0, 1)$ if and only if $p \geq 4/3$ and decreasing on $(0, 1)$ if and only if $p \leq 1$.*

Proof. Differentiation yields

$$(2.4) \quad F'_p(x) = \frac{x^{p-1}+1}{x^p+1} \frac{1}{(x-1) \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} \times f_p(x),$$

where

$$(2.5) \quad f_p(x) = \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1} - \sqrt{2} \frac{x-1}{(x+1)\sqrt{x^2+1}} \frac{x^p+1}{x^{p-1}+1}$$

Differentiating $f_p(x)$ and simplifying lead to

$$(2.6) \quad f'_p(x) = \frac{\sqrt{2}(1-x)x^p}{(\sqrt{x^2+1})^3(x+1)^2(x+x^p)^2} g(x),$$

where

$$(2.7) \quad g(x) = (x^{p+2} + x^{p+1} + 2x^p - x^{2-p} - x^{3-p} - 2x^{4-p} + (p-1)x^4 - x^3 + x - p + 1).$$

(i) We now prove that F_p is strictly increasing on $(0, 1)$ if and only if $p \geq 4/3$. From (2.4) it is seen that $\text{sgn } F'_p(x) = \text{sgn } f_p(x)$ for $x \in (0, 1)$, so it suffices to prove that $f_p(x) > 0$ for $x \in (0, 1)$ if and only if $p \geq 4/3$.

Necessity. If $f_p(x) > 0$ for $x \in (0, 1)$ then there must be $\lim_{x \rightarrow 1^-} (1-x)^{-3} f_p(x) \geq 0$. Application of L'Hospital rule leads to

$$\lim_{x \rightarrow 1^-} \frac{f_p(x)}{(1-x)^3} = \lim_{x \rightarrow 1^-} \frac{\ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1} - \sqrt{2} \frac{x-1}{(x+1)\sqrt{x^2+1}} \frac{x^p+1}{x^{p-1}+1}}{(1-x)^3} = \frac{1}{8} \left(p - \frac{4}{3} \right),$$

and so we have $p \geq 4/3$.

Sufficiency. We now prove $f_p(x) > 0$ for $x \in (0, 1)$ if $p \geq 4/3$. Since the Lehmer mean of order r of the positive real numbers a and b defined as

$$(2.8) \quad \mathcal{L}_r = \mathcal{L}_r(a, b) = \frac{a^{r+1} + b^{r+1}}{a^r + b^r}$$

(see [8]) is increasing in its parameter on \mathbb{R} , it is enough to show that $f_p(x) > 0$ for $x \in (0, 1)$ when $p = 4/3$. In this case, we have

$$g(x) = x - x^3 + \frac{1}{3}x^4 - x^{\frac{2}{3}} + 2x^{\frac{4}{3}} - x^{\frac{5}{3}} + x^{\frac{7}{3}} - 2x^{\frac{8}{3}} + x^{\frac{10}{3}} - \frac{1}{3},$$

and therefore

$$3g(x^3) = x^{12} + 3x^{10} - 3x^9 - 6x^8 + 3x^7 - 3x^5 + 6x^4 + 3x^3 - 3x^2 - 1.$$

Factoring yields that for $x \in (0, 1)$

$$3g(x^3) = (x-1)^3(x+1)(x^8 + 2x^7 + 7x^6 + 9x^5 + 9x^4 + 9x^3 + 7x^2 + 2x + 1) < 0.$$

It follows from (2.6) that $f'_p(x) < 0$, that is, the function f_p is decreasing on $(0, 1)$. Hence for $x \in (0, 1)$ we have $f_p(x) > f_p(1) = 0$, which proves the sufficiency.

(ii) We next prove that F_p is strictly decreasing on $(0, 1)$ if and only if $p \leq 1$. Similarly, it suffices to show that $f_p(x) < 0$ for $x \in (0, 1)$ if and only if $p \leq 1$.

Necessity. If $f_p(x) < 0$ for $x \in (0, 1)$ then we have

$$\lim_{x \rightarrow 0^+} f_p(x) = \begin{cases} \ln(\sqrt{2}-1) + \sqrt{2} & \text{if } p > 1 \\ \ln(\sqrt{2}-1) + \frac{\sqrt{2}}{2} & \text{if } p = 1 \\ \ln(\sqrt{2}-1) & \text{if } p < 1 \end{cases} \leq 0,$$

which yields $p \leq 1$.

Sufficiency. We prove $f_p(x) < 0$ for $x \in (0, 1)$ if $p \leq 1$. As mentioned previous, the function $p \mapsto L_{p-1}(1, x)$ is increasing on \mathbb{R} , it suffices to demonstrate $f_p(x) < 0$ for $x \in (0, 1)$ when $p = 1$. In this case, we have $g(x) = 2x - 2x^3 > 0$, then $f'_p(x) > 0$, and then for $x \in (0, 1)$ we have $f_p(x) < f_p(1) = 0$, which proves the sufficiency and the proof of this lemma is finished. \square

Lemma 3. *Let the function g be defined on $(0, 1)$ by (2.7). Then there is a unique $x_0 \in (0, 1)$ such that $g(x) < 0$ for $x \in (0, x_0)$ and $g(x) > 0$ for $x \in (x_0, 1)$ if $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \in (122/100, 4/3)$.*

Proof. We prove desired result stepwise.

Step 1: We have $g^{(4)}(x) > 0$ for $x \in (0, 1)$ when $p \in (1, 4/3)$.

Differentiations yield

$$(2.9) \quad g'(x) = (p+2)x^{p+1} + (p+1)x^p + 2px^{p-1} + (p-2)x^{1-p} + (p-3)x^{2-p} + 2(p-4)x^{3-p} + 4(p-1)x^3 - 3x^2 + 1,$$

$$(2.10) \quad g''(x) = (p+1)(p+2)x^p + p(p+1)x^{p-1} + 2p(p-1)x^{p-2} - (p-1)(p-2)x^{-p} - (p-2)(p-3)x^{1-p} - 2(p-3)(p-4)x^{2-p} + 12(p-1)x^2 - 6x,$$

$$\begin{aligned}
(2.11) g'''(x) &= p(p+1)(p+2)x^{p-1} + p(p-1)(p+1)x^{p-2} \\
&\quad + 2p(p-1)(p-2)x^{p-3} + p(p-1)(p-2)x^{-p-1} \\
&\quad + (p-1)(p-2)(p-3)x^{-p} + 2(p-2)(p-3)(p-4)x^{1-p} \\
&\quad + 24(p-1)x - 6,
\end{aligned}$$

$$\begin{aligned}
(2.12) \quad \frac{g^{(4)}(x)}{p-1} &= p(p+1)(p+2)x^{p-2} \\
&\quad + p(p+1)(p-2)x^{p-3} + 2p(p-2)(p-3)x^{p-4} \\
&\quad - p(p+1)(p-2)x^{-p-2} - p(p-2)(p-3)x^{-p-1} \\
&\quad - 2(p-2)(p-3)(p-4)x^{-p} + 24 \\
&: = I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= p(p+1)(p+2)x^{p-2} > 0, \\
I_2 &= p(p+1)(p-2)x^{p-3} + 2p(p-2)(p-3)x^{p-4} \\
&= p(2-p)x^{p-4}(2(3-p) - (p+1)x) \\
&> p(2-p)x^{p-3}(2(3-p) - (p+1)) = p(2-p)x^{p-3}(5-3p) > 0, \\
I_3 &= -p(p+1)(p-2)x^{-p-2} - p(p-2)(p-3)x^{-p-1} \\
&= p(2-p)x^{-p-2}((p+1) - (3-p)x) \\
&> p(2-p)x^{-p-2}((p+1) - (3-p)) = 2p(2-p)x^{-p-2}(p-1) > 0, \\
I_4 &= 2(2-p)(3-p)(4-p)x^{-p} + 24 > 0
\end{aligned}$$

Hence, $g^{(4)}(x) > 0$ for $x \in (0, 1)$ when $p \in (1, 4/3)$.

Step 2: There is unique $x_3 \in (0, 1)$ such that $g'''(x) < 0$ for $x \in (0, x_3)$ and $g'''(x) > 0$ for $x \in (x_3, 1)$ when $p \in (122/100, 4/3)$.

Since $g^{(4)}(x) > 0$ for $x \in (0, 1)$ when $p \in (1, 4/3)$, to prove this step, it suffices to verify that $g'''(0^+) < 0$ and $g'''(1) > 0$. Simple computation yields

$$\begin{aligned}
\operatorname{sgn} g'''(0^+) &= \operatorname{sgn}(p(p-1)(p-2)) < 0, \\
g'''(1) &= 8p^3 - 30p^2 + 94p - 84 := h(p) > 0,
\end{aligned}$$

where the last inequality holds is due to

$$h'(p) = 24p^2 - 60p + 94 = \frac{3}{2}(4p-5)^2 + \frac{113}{2} > 0$$

with

$$h\left(\frac{122}{100}\right) = \frac{17337}{31250} > 0 \quad \text{and} \quad h\left(\frac{4}{3}\right) = \frac{188}{27} > 0.$$

Step 3: There is a unique $x_2 \in (0, x_3)$ such that $g''(x) > 0$ for $x \in (0, x_2)$ and $g''(x) < 0$ for $x \in (x_2, 1)$ when $p \in (122/100, 4/3)$.

By Step 2 with

$$\begin{aligned}
\operatorname{sgn} g''(0^+) &= \operatorname{sgn}(-(p-1)(p-2)) > 0, \\
g''(1) &= 12(3p-4) < 0,
\end{aligned}$$

we see that $g''(x) < g''(1) < 0$ for $x \in (x_3, 1)$ but $g''(0^+) > 0$, which completes this step.

Step 4: There are two $x_{11} \in (0, x_2), x_{12} \in (x_2, 1)$ such that $g'(x) < 0$ for $x \in (0, x_{11}) \cup (x_{12}, 1)$ and $g'(x) > 0$ for $x \in (x_{11}, x_{12})$ when $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \in (\frac{122}{100}, \frac{4}{3})$.

Since $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \in (\frac{122}{100}, \frac{4}{3})$, according to Step 3 and note that

$$\begin{aligned} g'(0^+) &= \operatorname{sgn}(p-2) < 0, \\ g'(1) &= 4(3p-4) < 0, \end{aligned}$$

in order to prove this step, it is enough to verify that $g'(x_2) > 0$.

In fact, if $g'(x_2) < 0$ then $g'(x) < g'(x_2) < 0$ for $x \in (0, x_2)$ and $g'(x) < g'(x_2) < 0$ for $x \in (x_2, 1)$, and then $g'(x) < 0$ for $x \in (0, 1)$. It follows that $g(x) > g(1) = 0$, which in combination (2.6) yields $f'_p(x) > 0$. Therefore, $f_p(x) < f_p(1) = 0$, which implies from (2.4) that $F'_p(x) < 0$. Then for $x \in (0, 1)$

$$0 = F_p(0^+) > F_p(x) > F_p(1) = 0$$

if $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \in (\frac{122}{100}, \frac{4}{3})$, which is clearly a contradiction. Hence there must be $g'(x_2) > 0$, which completes the Step 4.

Step 5: There is a unique $x_0 \in (x_{11}, x_{12})$ such that $g(x) < 0$ for $x \in (0, x_0)$ and $g(x) > 0$ for $x \in (x_0, 1)$ if $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})} \in (122/100, 4/3)$.

From Step 5 and notice that

$$g(0^+) = 1 - p < 0, \quad g(1^-) = 0,$$

we have the following variance table of $g(x)$:

x	0^+	$(0, x_{11})$	x_{11}	(x_{11}, x_{12})	x_{12}	$(x_{12}, 1)$	1
$g'(x)$	—	—	0	+	0	—	—
$g(x)$	—	\searrow	—	\nearrow	+	\searrow	0

where

$$g(x_{11}) < g(0^+) = 1 - p < 0 \quad \text{and} \quad g(x_{12}) > g(1) = 0.$$

Thus the step follows. \square

Lemma 4. Let the function f_p be defined on $(0, 1)$ by (2.5), where $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$. Then there is a unique $\tilde{x}_0 \in (0, x_0)$ to satisfy $f_p(\tilde{x}_0) = 0$ such that $f_p(x) > 0$ for $x \in (0, \tilde{x}_0)$ and $f_p(x) < 0$ for $x \in (\tilde{x}_0, 1)$.

Proof. Due to (2.6), it is deduced that f_p is decreasing on $(0, x_0)$ and increasing on $(x_0, 1)$, then $f_p(x) < f_p(1) = 0$ for $x \in (x_0, 1)$ but $f_p(0^+) = \ln(\sqrt{2}-1) + \sqrt{2} > 0$. This indicates that there is a unique $\tilde{x}_0 \in (0, x_0)$ to satisfy $f_p(\tilde{x}_0) = 0$ such that $f_p(x) > 0$ for $x \in (0, \tilde{x}_0)$ and $f_p(x) < 0$ for $x \in (\tilde{x}_0, 1)$. \square

3. PROOFS OF MAIN RESULTS

Based on the lemmas in the above section, we can easily proved our main results.

Proof of Theorem 1. By symmetry, we assume that $a > b > 0$. Then inequality $N < A_p$ is equivalent to

$$(3.1) \quad \ln N(1, x) - \ln A_p(1, x) = F_p(x) < 0,$$

where $x = b/a \in (0, 1)$. Now we prove the inequality (3.1) holds for all $x \in (0, 1)$ if and only if $p \geq 4/3$.

Necessity. If inequality (3.1) holds, then by Lemma 1 we have

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-4) \leq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \frac{1}{p} \ln 2 - \ln \ln(3+2\sqrt{2}) \leq 0 \text{ if } p > 0, \end{cases}$$

which yields $p \geq 4/3$.

Sufficiency. Suppose that $p \geq 4/3$. It follows from Lemma 2 that $F_p(x) < F_p(1) = 0$ for $x \in (0, 1)$, which proves the sufficiency.

Using the monotonicity of the function $x \mapsto F_{4/3}(x)$ on $(0, 1)$, we have

$$\ln \frac{1}{\sqrt[4]{2} \ln(\sqrt{2}+1)} = F_{4/3}(0^+) < F_{54/3}(x) < F_{4/3}(1^-) = 0,$$

which implies (1.25).

Thus the proof of Theorem 1 is finished. \square

Proof of Theorem 2. Clearly, the inequality $N > A_p$ is equivalent to

$$(3.2) \quad \ln N(1, x) - \ln A_p(1, x) = F_p(x) > 0,$$

where $x = b/a \in (0, 1)$. Now we show that the inequality (3.2) holds for all $x \in (0, 1)$ if and only if $p \leq \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$.

Necessity. The condition $p \leq \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$ is necessary. Indeed, if inequality (3.2) holds, then we have

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-4) \geq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \frac{1}{p} \ln 2 - \ln \ln(3+2\sqrt{2}) \geq 0 \text{ if } p > 0 \end{cases}$$

or

$$\begin{cases} \lim_{x \rightarrow 1^-} \frac{F_p(x)}{(x-1)^2} = -\frac{1}{24}(3p-4) \geq 0, \\ \lim_{x \rightarrow 0^+} F_p(x) = \infty \text{ if } p \leq 0. \end{cases}$$

Solving the above inequalities leads to $p \leq \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$.

Sufficiency. The condition $p \leq \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$ is also sufficient. Since the function $r \mapsto A_r(1, x)$ is increasing, so the function $p \mapsto F_p(x)$ is decreasing, thus it is suffices to show that $F_p(x) > 0$ for all $x \in (0, 1)$ if $p = p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$.

Lemma 4 reveals that for $p = p_0$ there is a unique \tilde{x}_0 to satisfy

$$(3.3) \quad f_{p_0}(x) = \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1} - \sqrt{2} \frac{x-1}{(x+1)\sqrt{x^2+1}} \frac{x^{p_0}+1}{x^{p_0-1}+1} = 0$$

such that the function $x \mapsto F_p(x)$ is strictly increasing on $(0, \tilde{x}_0)$ and strictly decreasing on $(\tilde{x}_0, 1)$. It is acquired that for $p_0 = \frac{\ln 2}{\ln \ln(3+2\sqrt{2})}$

$$\begin{aligned} 0 &= F_{p_0}(0^+) < F_{p_0}(x) \leq F_{p_0}(\tilde{x}_0) \\ 0 &= F_{p_0}(1) < F_{p_0}(x_3) \leq F_{p_0}(\tilde{x}_0), \end{aligned}$$

which leads to

$$A_{p_0}(1, x) < N(1, x) < (\exp F_p(\tilde{x}_0)) A_{p_0}(1, x).$$

Solving the equation (3.3) by mathematical computation software we find that $\tilde{x}_0 \in (0.15806215485976, 0.15806215485977)$, and then

$$\beta_2 = \exp F_p(\tilde{x}_0) \approx 1.0138,$$

which proves the sufficiency and inequalities (1.25). \square

4. COROLLARIES

From the proof of Lemma 2, it is seen that $f_p(x) > 0$ for $x \in (0, 1)$ if and only if $p \geq 4/3$, which implies that the inequality

$$N(1, x) = \frac{x-1}{2 \ln \frac{x-1+\sqrt{2(x^2+1)}}{x+1}} > \frac{x+1}{2} \sqrt{\frac{x^2+1}{2}} \frac{x^{p-1}+1}{x^p+1}$$

holds if and only $p \geq 4/3$. In a similar way, the inequality

$$N(1, x) < \frac{x+1}{2} \sqrt{\frac{x^2+1}{2}} \frac{x^{p-1}+1}{x^p+1}$$

is valid if and only if $p \leq 1$. The results can be restated as a corollary.

Corollary 1. *The inequalities*

$$(4.1) \quad \frac{AA_2}{\mathcal{L}_{p-1}} < N < \frac{AA_2}{\mathcal{L}_{q-1}}$$

holds if and only if $p \geq 4/3$ and $q \leq 1$, where \mathcal{L}_r is the Lehmer mean defined by (2.8).

Using the monotonicity of the function defined on $(0, 1)$ by

$$F_p(x) = \ln \frac{N(1, x)}{A_p(1, x)}$$

given in Lemma 2, we can obtain a Fan Ky type inequality but omit the further details of the proof.

Corollary 2. *Let $a_1, a_2, b_1, b_2 > 0$ with $a_1/b_1 < a_2/b_2 < 1$. Then the following Fan Ky type inequality*

$$\frac{N(a_1, b_1)}{N(a_2, b_2)} < \frac{A_p(a_1, b_1)}{A_p(a_2, b_2)}$$

holds if $p \geq 4/3$. It is reversed if $p \leq 1$.

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